APPROXIMATION PROCEDURE FOR BOUNDARY-VALUE PROBLEMS OF HEAT CONDUCTION

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An approximation procedure is presented based on the method of moments which significantly reduces the amount of mathematical calculations and formalizes the selection of approximating differential equations.

The approximation of replacing a boundary-value problem of nonstationary heat conduction by a system of ordinary differential equations or transfer functions for each point of the body under study is a widely used technique in engineering calculations. An effective method of generating these approximations is the method of moments [1-3], which gives good approximate results in wide time and frequency ranges with the use of simple relations. However, its application usually requires laborious mathematical manipulations. In addition, in order to choose the approximating differential equations (or transfer functions), the exact forms of the time or frequency characteristics must be known at the given point of the body.

The most difficult step in this approximation is finding the linear integral values l_i. By definition [3]

$$l_i = \int_0^\infty \int_0^t \ldots \int_0^t (\varphi_{i-1} - \varphi)\, dt^i, \quad i = 1, 2, \ldots \tag{1}$$

In analytical approximations expression (1) is not directly used because it requires the previous calculation of $\varphi(t)$. The l_i are obtained from the transfer function Y(r, s) of the body, relating the Laplace transform of the gain $\Theta(r, t) - \Theta(r, 0)$ and that of the excitation $U(t) - U(0)$ for an initial distribution $\Theta(r, 0)$. The transfer function Y(r, s) is obtained by solving the original boundary-value problem for the Laplace transform [4]. The l_i are then determined from the relation [1]

$$l_i = (-1)^i \frac{1}{i!} \lim_{s\to 0} \frac{d^i Y(r, s)}{ds^i}. \tag{2}$$

The calculation of l_2 from (2) for multilayered walls or even for a single-layer cylinder requires a great deal of mathematical labor. It is more sensible to determine the l_i from an expansion of Y(r, s) in a power series in s [3]:

$$Y(r, s) = K\left(1 + \sum_{i=1}^{\infty} a_i s^i\right), \tag{3}$$

and then

$$l_i = Ka_i. \tag{4}$$

Relation (4) was used in [2, 5]. However, it is still very laborious to obtain (3) for cylindrical bodies.

The approximation procedure discussed here is based on the expansion of various combinations of transcendental functions typical for the transfer functions of bodies having the form of an infinite cylinder, infinite plate, or a sphere. In the analysis of the transfer functions of single-layer bodies of the above forms for various combinations of boundary conditions of the first and second kinds, it has been established that for any type of boundary condition and independently of the number of layers, the transfer functions for cylindrical walls can be written in terms of three combinations of modified Bessel functions

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$$\omega_1^c[a,\ b] = I_0(az)K_0(bz) - I_0(bz)K_0(az),$$

$$\omega_2^c[a,\ b] = I_0(az)K_1(bz) + I_1(bz)K_0(az),$$

$$\omega_3^c[a,\ b] = I_1(az)K_1(bz) - I_1(bz)K_1(az),$$

where a and b are positive numbers and $z = \sqrt{s/a'}$. We also have

$$\omega_1^c[a,\ b] = -\omega_1^c[b,\ a],\ \omega_3^c[a,\ b] = -\omega_3^c[b,\ a].$$

For solid cylinders the structure of the transfer function involves only the two forms

$$\omega_2^c[a,\ 0] = \frac{1}{z}I_0(az)\ \text{и}\ \omega_3^c[0,\ b] = -\frac{1}{z}I_1(bz).$$

For plane walls the transfer function is characterized by the following two combinations of hyperbolic functions:

$$\omega_1^p[a,\ b] = \omega_3^p[a,\ b] = \frac{1}{z}[\mathrm{sh}(az)\,\mathrm{ch}(bz) - \mathrm{sh}(bz)\,\mathrm{ch}(az)] = \frac{1}{z}\mathrm{sh}[(a-b)z],$$

$$\omega_2^p[a,\ b] = \frac{1}{z}[\mathrm{ch}(az)\,\mathrm{ch}(bz) - \mathrm{sh}(az)\,\mathrm{sh}(bz)] = \frac{1}{z}\mathrm{ch}[(a-b)z],$$

$$\omega_{1,3}^p[a,\ b] = -\omega_{1,3}^p[b,\ a],\ \omega_2^p[a,\ b] = \omega_2^p[b,\ a].$$

The transfer function of a spherical shell can be expressed in terms of

$$\omega_1^s[a,\ b] = \frac{1}{ab}\frac{1}{z}\mathrm{sh}[(a-b)z],$$

$$\omega_2^s[a,\ b] = \frac{1}{ab}\frac{1}{z}\left\{\frac{1}{bz}\mathrm{sh}[(a-b)z] + \mathrm{ch}[(a-b)z]\right\},$$

$$\omega_3^s[a,\ b] = \frac{1}{ab}\frac{1}{z}\left\{\left(1 - \frac{1}{ab}\frac{1}{z^2}\right)\mathrm{sh}[(a-b)z] - \frac{1}{z}\left(\frac{1}{a} - \frac{1}{b}\right)\mathrm{ch}[(a-b)z]\right\},$$

$$\omega_1^s[a,\ b] = -\omega_1^s[b,\ a],\ \omega_3^s[a,\ b] = -\omega_3^s[b,\ a]. \tag{5}$$

For a solid sphere we have

$$\omega_2^s[a,\ 0] = \frac{1}{a}\frac{1}{z^2}\mathrm{sh}(az);\ \omega_3^s[0,\ b] = \frac{1}{b}\frac{1}{z^2}\left[\frac{1}{bz}\mathrm{sh}(bz) - \mathrm{ch}(bz)\right].$$

We obtain series expansions for these combinations as follows:

$$\omega_1[a,\ b] = \alpha_0[a,\ b] + \alpha_1[a,\ b]z^2 + \alpha_2[a,\ b]z^4 + O(z^4),$$

$$\omega_2[a,\ b] = \mu_0[a,\ b]\frac{1}{z} + \mu_1[a,\ b]z + \mu_2[a,\ b]z^3 + O(z^3), \tag{6}$$

$$\omega_3[a,\ b] = \nu_0[a,\ b] + \nu_1[a,\ b]z^2 + \nu_2[a,\ b]z^4 + O(z^4),$$

where for cylindrical walls [6]

$$\alpha_0^c[a,\ b] = \ln\frac{a}{b},\ \alpha_1^c[a,\ b] = \frac{1}{4}\left[(a^2+b^2)\ln\frac{a}{b} + b^2 - a^2\right],$$

$$\alpha_2^c[a,\ b] = \frac{a^4 + 4a^2b^2 + b^4}{64}\ln\frac{a}{b} + \frac{3}{128}(b^4 - a^4),$$

$$\mu_0^c[a,\ b] = \frac{1}{b},\ \mu_1^c[a,\ b] = \frac{b}{2}\ln\frac{b}{a} + \frac{a^2 - b^2}{4b},$$

$$\mu_2^c[a,\ b] = \frac{1}{16}(b^3 + 2a^2b)\ln\frac{b}{a} + \frac{(a^2 - b^2)(a^2 + 5b^2)}{64b},$$

$$\nu_0^c[a,\ b] = \frac{a^2 - b^2}{2ab},\ \nu_1^c[a,\ b] = \frac{ab}{4}\ln\frac{b}{a} + \frac{a^4 - b^4}{16ab},$$

$$\nu_2^c[a,\ b] = \frac{1}{32}\left[ab(a^2 + b^2)\ln\frac{b}{a} + \frac{3}{4}ab(a^2 - b^2) + \frac{a^6 - b^6}{12ab}\right],$$

for solid cylinders

$$\mu_0^c[a,\ 0]=1,\ \mu_1^c[a,\ 0]=\frac{a^2}{4},\ \mu_2^c[a,\ 0]=\frac{a^4}{64},$$

$$\nu_0^c[0,\ b]=-\frac{b}{2},\ \nu_1^c[0,\ b]=-\frac{b^3}{16},\ \nu_2^c[0,\ b]=-\frac{b^5}{384},$$

for plane walls

$$\alpha_0^p[a,\ b]=a-b,\ \alpha_1^p[a,\ b]=\frac{1}{6}(a-b)^3,\ \alpha_2^p[a,\ b]=\frac{1}{120}(a-b)^5,$$

$$\mu_0^p[a,\ b]=1,\ \mu_1^p[a,\ b]=\frac{1}{2}(a-b)^2,\ \mu_2^p[a,\ b]=\frac{1}{24}(a-b)^4,$$

for spherical shells

$$\alpha_0^s[a,\ b]=\frac{a-b}{ab},\ \alpha_1^s[a,\ b]=\frac{(a-b)^3}{6ab},\ \alpha_2^s[a,\ b]=\frac{(a-b)^5}{120ab},$$

$$\mu_0^s[a,\ b]=\frac{1}{b^2},\ \mu_1^s[a,\ b]=\frac{(a+2b)(a-b)^2}{6ab^2},$$

$$\mu_2^s[a,\ b]=\frac{(a+4b)(a-b)^4}{120ab^2},$$

$$\nu_0^s[a,\ b]=\frac{(a-b)[(a-b)^2+3ab]}{3a^2b^2},\ \nu_1^s[a,\ b]=\frac{(a-b)^3[(a-b)^2+5ab]}{30a^2b^2},$$

$$\nu_2^s[a,\ b]=\frac{(a-b)^5[(a-b)^2+7ab]}{840a^2b^2},$$

and for solid spheres

$$\mu_0^s[a,\ 0]=1,\ \mu_1^s[a,\ 0]=\frac{a^2}{6},\ \mu_2^s[a,\ 0]=\frac{a^4}{120},$$

$$\nu_0^s[0,\ b]=-\frac{b}{3},\ \nu_1^s[0,\ b]=-\frac{b^3}{30},\ \nu_2^s[0,\ b]=-\frac{b^5}{840}.$$

$O(\alpha)$ in (6) denotes an infinitesimal quantity of higher order than $\alpha = \alpha(z)$ in the limit $z \to 0$.

With the help of the above combinations and their expansions, $Y(r,\ s)$ can easily be put in the form of a ratio of two power series:

$$Y(r,\ s)=\frac{d_0+d_1z^2+d_2z^4+\ldots}{c_0+c_1z^2+c_2z^4+\ldots} \tag{7}$$

Then we have the relations

$$l_0=\lim_{s\to 0}Y(r,\ s)=\frac{d_0}{c_0},\ l_1=\lim_{s\to 0}\frac{d}{ds}Y(r,\ s)=\frac{d_1c_0-d_0c_1}{a'c_0^2},$$

$$l_2=\lim_{s\to 0}\frac{d^2}{ds^2}Y(r,\ s)=\frac{2}{(a')^2}\left[\frac{d_2c_0-d_0c_2}{c_0^2}-\frac{c_1(d_1c_0-d_0c_1)}{c_0^3}\right].$$

Analysis of the transient and frequency characteristics of bodies with the geometrical forms considered above, and for various combinations of boundary conditions of the first, second, third, and fourth kinds has shown that for fixed r they can be described (with sufficient accuracy for the solution of many practical problems) by the following transfer functions of circuits with lumped parameters:

$$W_1(s)=\frac{K}{T_1T_2s^2+(T_1+T_2)s+1},\ W_2(s)=\frac{K}{T_4s+1}\exp(-\tau s),$$

$$W_3(s) = K\frac{T_3 s + 1}{T_4 s + 1},\quad W_4(s) = K[T_5T_6s^2 + (T_5 + T_6)s + 1] \tag{8}$$

or by the corresponding ordinary differential equations.

It has been established that the quantities T_1T_2 and $T_1 + T_2$ can be used to uniquely choose an approximating transfer function from the set (8) (in the absense of exact dynamical characteristics).

With these results and a comparatively small amount of elementary calculation, we can approximately replace a boundary-value problem of nonstationary heat conduction by a system of differential equations corresponding to the transfer functions of the forms in (8), along with the initial conditions of the original problem. The following is the procedure [7]:

1. Solve the boundary-value problem for the Laplace transform and thereby obtain Y(r, s).

2. Decompose Y(r, s) into the combinations $\omega_i[a, b]$.

3. Put Y(r, s) in the form (7) with the help of $\alpha_k[a, b]$, $\mu_k[a, b]$, and $\nu_k[a, b]$ (k = 0, 1, 2).

4. Calculate for a given $r = r_j$

$$K = \frac{d_0}{c_0},\quad T_1 + T_2 = \frac{1}{a'}\left(\frac{c_1}{c_0} - \frac{d_1}{d_0}\right),\quad T_1T_2 = (T_1 + T_2)^2 - L, \tag{9}$$

$$L = \frac{1}{(a')^2}\left(\frac{d_2}{d_0} - \frac{c_2}{c_0} - \frac{c_1d_1}{c_0d_0} + \frac{c_1^2}{c_0^2}\right).$$

5. Choose for the given $r = r_j$ the type of approximating transfer function and determine its parameters:

a) if $T_1 + T_2 \geqslant 0$ and $T_1T_2 \geqslant 0$, then $Y(r_j, s) \approx W_1(s)$ or $W_2(s)$, where $T_4 = L/(T_1 + T_2)$, $\tau = -T_4 \ln[L/(T_1 + T_2)^2]$, and if $\xi = (T_1 + T_2)/2\sqrt{T_1T_2} \geqslant 1$, then

$$W_1(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)},\quad T_{1,2} = \frac{T_1 + T_2}{2} \pm \sqrt{\frac{(T_1 + T_2)^2}{4} - T_1T_2},$$

otherwise

$$W_1(s) = \frac{K}{T^2s^2 + 2\xi Ts + 1},\quad T = \sqrt{T_1T_2};$$

b) if $T_1 + T_2 \geqslant 0$ and $T_1T_2 < 0$, then $Y(r_j, s) \approx W_3(s)$, where $T_4 = L/(T_1 + T_2)$, $T_3 = T_4 - (T_1 + T_2)$;

c) if $T_1 + T_2 < 0$, $T_1T_2 > 0$ and $L > 0$, then $Y(r_j, s) \approx W_4(s)$, where $T_5T_6 = L$, $T_5 + T_6 = -(T_1 + T_2)$, and if

$$\xi_1 = \frac{T_5 + T_6}{2\sqrt{T_5T_6}} \geqslant 1,$$

then

$$W_4(s) = K(T_5 s + 1)(T_6 s + 1),\quad T_{5,6} = \frac{T_5 + T_6}{2} \pm \sqrt{\frac{(T_5 + T_6)^2}{4} - T_5T_6},$$

otherwise

$$W_4(s) = K(T_д^2 s^2 + 2\xi_1 T_д s + 1),\quad T_д = \sqrt{T_5T_6};$$

d) if $T_1 + T_2 < 0$, $T_1T_2 > 0$ and $L < 0$, then $Y(r_j, s) \approx W_3(s)$.

The transfer function of a body with N layers of different thermal diffusivities contains N variables $z_1 = \sqrt{s/a_1'}, \ldots, z_N = \sqrt{s/a_N'}$. Therefore, before decomposing Y(r, s) into the combinations $\omega_i[a, b]$ it is necessary to transfor to a single variable z (say $z = z_1$) using the relation $z_n = \varepsilon_n z$, where $\varepsilon_n = \sqrt{a_1'/a_n'}$, n = 1, 2,...,N. Obviously in this case one must take $a' = a_1'$ in using (9), and the coefficients a and b in the combinations $\omega_i[a, b]$ and their expansions must take into account the ε_n. For example,

$$I_0(az_2)K_0(bz_2) - I_0(bz_2)K_0(az_2) = I_0(a\varepsilon_2 z)K_0(b\varepsilon_2 z) - I_0(b\varepsilon_2 z)K_0(a\varepsilon_2 z) = \omega_1^c[a\varepsilon_2,\ b\varepsilon_2],$$

$$\alpha_1^c[a\varepsilon_2,\ b\varepsilon_2] = \frac{1}{4}\left\{[(a\varepsilon_2)^2 + (b\varepsilon_2)^2]\ln\frac{a}{b} + (b\varepsilon_2)^2 - (a\varepsilon_2)^2\right\}.$$

We note that for given boundary conditions the form of Y(r, s) in terms of the $\omega_i[a, b]$ and the expressions for the coefficients c_k, d_k in terms of $\alpha_k[a, b]$, $\mu_k[a, b]$, and $\nu_k[a, b]$ are identical for bodies of the geometrical forms considered here. This property is especially useful in choosing the simplest computational scheme.

Our procedure was tested for single-layer and double-layer cylindrical plates and spherical shells for various possible combinations of the different types of boundary conditions by comparing with the exact frequency characteristics, Y(r, s), and transient characteristics obtained by solving the original boundary-value problems by the grid method. The application of our procedure to diverse control optimization problems of thermal processes, thermal instrumentation and temperature converters shows that the procedure is highly effective.

Example. It is required to approximate the following boundary-value problem for a complete cylinder:

$$\begin{gathered}\frac{\partial\Theta(r,\ t)}{\partial t} = a'\left[\frac{\partial^2\Theta(r,\ t)}{\partial r^2} + \frac{1}{r}\frac{\partial\Theta(r,\ t)}{\partial r}\right],\ t>0,\ r\in[R_1,\ R_2],\\ -\lambda\frac{\partial\Theta(R_1,\ t)}{\partial r} = 0,\quad -\lambda\frac{\partial\Theta(R_2,\ t)}{\partial r} + q(t) = 0,\\ \Theta(r,\ 0) = 0.\end{gathered} \tag{10}$$

In this case U(t) — U(0) = q(t) (q(0) = 0). After taking the Laplace transform of (10) [4] we obtain the solution for the transform:

$$\Theta(r,\ s) - \frac{1}{s}\Theta(r,\ 0) = Y(r,\ s)\,q(s),\ z = \sqrt{s/a'},$$

where

$$Y(r,\ s) = \frac{I_0(rz)K_1(R_1z) + I_1(R_1z)K_0(rz)}{\lambda z\,[I_1(R_1z)K_1(R_2z) - I_1(R_2z)K_1(R_1z)]} = \frac{\omega_2^c[r,\ R_1]}{\lambda z\omega_3^c[R_1,\ R_2]}.$$

This transfer function cannot be represented in the form (7) directly. Therefore, we multiply numerator and denominator of Y(r, s) by z and obtain

$$Y(r,\ s) = \frac{a'}{\lambda}\frac{1}{s}Y^*(r,\ s),\ Y^*(r,\ s) = \frac{z\omega_2^c[r,\ R_1]}{\omega_3^c[R_1,\ R_2]}.$$

It follows from (6) that

$$Y^*(r,\ s) = \frac{\mu_0^c[r,\ R_1] + \mu_1^c[r,\ R_1]z^2 + \mu_2^c[r,\ R_1]z^4 + \ldots}{\nu_0^c[R_1,\ R_2] + \nu_1^c[R_1,\ R_2]z^2 + \nu_2^c[R_1,\ R_2]z^4 + \ldots}$$

Therefore,

$$d_0 = \mu_0^c[r,\ R_1],\ d_1 = \mu_1^c[r,\ R_1],\ d_2 = \mu_2^c[r,\ R_1],$$

$$c_0 = \nu_0^c[R_1,\ R_2],\ c_1 = \nu_1^c[R_1,\ R_2],\ c_2 = \nu_2^c[R_1,\ R_2].$$

We illustrate the procedure for the example: R_1 = 0.02 m, R_2 = 0.1 m, a' = $1.35\cdot10^{-5}$ m^2/sec, λ = 51.5 W/(m·°K).

For r = 0.1 m

$$K = -20{,}83\ \text{м}^{-1},\quad T_1 + T_2 = -8247\ \text{sec},\quad T_1T_2 = 8937\ \text{sec}^2,\quad L = -2135\ \text{sec}^2,$$

$$Y^*(0{,}1;\ s) \approx W_3(s) = K\frac{T_3s+1}{T_4s+1},\ T_3 = 108{,}4\ \text{sec}, T_4 = 25{,}88\ \text{sec}$$

For r = 0.05 m

TABLE 1. Solution of the Boundary Problem (10) by the Grid Method and by Approximating Differential Equations

$Fo = ta'/R^2$		0	0,010	0,020	0,050	0,100	0,150	0,200	0,275	0,375
$\Theta(0,1;\ t)$	grid method	0	2,152	3,205	5,385	8,067	10,35	12,48	15,56	19,62
°C	from (15)	0	1,525	2,772	5,449	8,291	10,51	12,58	15,62	19,67
$\Theta(0,05;\ t)$	grid method	0	0,001	0,019	0,434	1,868	3,656	5,588	8,574	12,61
°C	from (16)	0	0,009	0,057	0,496	1,899	3,688	5,622	8,615	12,65

Note. When Fo → ∞, the approximation errors decrease.

$$K = -20{,}83\ \text{m}^{-1},\quad T_1 + T_2 = 46{,}15\ \text{sec},\ T_1T_2 = 314{,}2\ \text{sec}^2,\ L = 1815\ \text{sec}^2,$$

$$Y^*(0{,}05;\ s) \approx W_1(s) = \frac{K}{(T_1s+1)(T_2s+1)},\quad T_1 = 37{,}84\ \text{sec},\ T_2 = 8{,}303\ \text{sec},$$

Therefore,

$$\Theta(0{,}1;\ s) - \frac{1}{s}\,\Theta(0{,}1;\ 0) \approx \frac{a'}{\lambda}\,\frac{1}{s}\,K\,\frac{T_3s+1}{T_4s+1}\,q(s), \tag{11}$$

$$\Theta(0{,}05;\ s) - \frac{1}{s}\,\Theta(0{,}05;\ 0) \approx \frac{a'}{\lambda}\,\frac{1}{s}\,\frac{K}{(T_1s+1)(T_2s+1)}\,q(s). \tag{12}$$

Hence for the values of r considered above, the boundary-value problem (10) can be replaced approximately by a system of ordinary differential equations with corresponding initial conditions

$$T_4\frac{d^2\Theta(t)}{dt^2} + \frac{d\Theta(t)}{dt} = \frac{a'}{\lambda}K\left[T_3\frac{dq(t)}{dt} + 1\right],\quad r = 0{,}1\ \text{m},$$
$$\Theta(0{,}1;\ 0) = 0,\quad q(0) = 0, \tag{13}$$

$$T_1T_2\frac{d^3\Theta(t)}{dt^3} + (T_1+T_2)\frac{d^2\Theta(t)}{dt^2} + \frac{d\Theta(t)}{dt} = \frac{a'}{\lambda}Kq(t),\quad r = 0{,}05\ \text{m},$$
$$\Theta(0{,}05;\ 0) = 0,\quad q(0) = 0. \tag{14}$$

The accuracy of the approximation can be estimated by comparing the transient characteristics obtained with the help of (13) and (14) with the transient behavior obtained from the solution of (10) by the grid method. We assume that at $t = 0^+$, q gradually varies from 0 to q_1. The solutions of (13) and (14) are easily obtained by operational methods using (11) and (12), Since in this example we have $q(s) = q_1/s$:

$$\Theta(0{,}1;\ t) \approx -\frac{a'}{\lambda}Kq_1\left[(T_4 - T_3)\exp\left(-\frac{t}{T_4}\right) - T_4 + T_3 + t\right], \tag{15}$$

$$\Theta(0{,}05;\ t) \approx -\frac{a'}{\lambda}Kq_1\left[\frac{T_1^2}{T_1-T_2}\exp\left(-\frac{t}{T_1}\right) - \frac{T_2^2}{T_1-T_2}\exp\left(-\frac{t}{T_2}\right) - T_1 - T_2 + t\right]. \tag{16}$$

The results of the comparison are given in Table 1 for the case $q_1 = 10{,}000\ \text{W/m}^2$. The results can also be compared with the analytical solution given in [4].

NOTATION

φ, φ_{i-1}, approximated and (i — 1)st approximating transient characteristics; $\Theta(r, t)$, temperature field; r, t, coordinate and time; U(t), the perturbation causing the variation $\Theta(r, t) - \Theta(r, 0)$; s, Laplace transform variable; K, static gain coefficient; $T_1, \ldots, T_6$, time constants; τ, lag; a' and λ, thermal diffusivity and thermal conductivity.

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FINITE-ELEMENT MODELS FOR CALCULATING THE TEMPERATURE FIELDS OF UNDERGROUND PIPELINES

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A simple method of constructing finite elements for the numerical modeling of temperature fields of underground pipelines is outlined.

In the practical design of mains pipelines, problems of temperature-field calculation are of particular interest [1]. Variations in the temperature conditions of the pipeline and the surrounding material exert an influence on the stress—strain state of the tube and cause settling of the earth, which leads to stability loss of the pipeline and other undesirable consequences [2]. Recently, there has been a trend to more completely taking account of the whole set of real conditions of pipeline use. In connection with this, there has been a significant increase in the role of numerical methods of calculation with the use of a computer. The most flexible and universal method is the finite-element method (FEM), which presently occupies the central position in engineering calculations [3]. FEM allows the thermal interaction of the pipeline with surrounding material of inhomogeneous structure to be analyzed [4], and allows the influence of imperfections and damage in the insulation and other anomalies in the thermophysical characteristics to be taken into account.

The present work outlines a simplified method of constructing finite elements (FE), taking account of the geometry of the problem and allowing errors of boundary approximation to be eliminated. In this case, plane FE in polar coordinates (Fig. 1) and three-dimensional FE in cylindrical coordinates (Fig. 2) are most appropriate.

Constructing an interpolational polynomial for the FE entails selecting an appropriate system of finite basis functions. The temperature values T_i at corners of the FE are taken as the interpolation parameters. The problem reduces to constructing the coordinate functions Φ_i. Usually [3], this problem involves investigating a system of linear algebraic equations, the order of which is equal to the number of degrees of freedom of the FE. Solving such systems by means of matrix algebra entails laborious transformations and well-known calculational difficulties.

A simple and expedient method of geometric formalization of the FE basis is now written [5], generalizing the probability concept of baricentric coordinates of simplex models. Probability identification of the basis significantly simplifies the procedure for constructing

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